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Two Benchmarks to Assess Two-Dimensional Theories of Sandwich, Composite Plates

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Two-dimensional theories and finite elements are assessed to analyze displacement and stress fields in sandwich, composites plates. Two benchmarks are used to conduct the assessment. The first benchmark is a sandwich plate loaded by harmonic distribution of transverse pressure for which a three-dimensional closed-form solution exists in the literature. The second benchmark is a rectangular sandwich plate loaded by a transverse pressure located at the center. More than 20 plate theories and finite elements were implemented in a unified formulation recently proposed by the authors. Classical theories based on displacement assumptions are compared to advanced mixed models formulated on the basis of Reissner's mixed variational theorem. Both equivalent single-layer models as well as layerwise models are considered. Analytical closed-form solutions and finite elements are given. The considered benchmarks highlight both the performance and limitations of the considered two-dimensional theories. The convenience of layerwise description and advanced mixed theories has been demonstrated. The second benchmark especially proved the need for layerwise models to capture the local effects.

Nomenclature

a, b, h	=	plate/shell geometrical parameters (length, width, and thickness)
N	=	order of the expansions used for transverse stresses and displacements
N_l	=	number of constituent layers of multilayered plate/shell
V	=	plate volume
x, y, z	=	Cartesian coordinates reference systems used for plates
Ω	=	plate reference surface

Subscript and Superscripts

a	=	results related to closed-form solutions
k	=	parameters related to the k layer

Introduction

SANDWICH structures combine light with height stiffness, high structural efficiency, and durability and, therefore, have been widely used to build large portions of aerospace, automotive, and ship vehicles. The expanded application of fibrous composite materials has included the use of laminated composites as face sheets for sandwich structures. The high performance of these composite materials makes them ideal candidates for use in future high-speed aircraft, spacecraft, satellite antennas, and reflectors of terrestrial systems.

This paper focuses on the mechanical modeling of sandwich plates constituted by layered composite faces. It addresses both full three-dimensional and two-dimensional descriptions. (Three-dimensional analysis is employed to model both skins and core, and two-dimensional plates theories are used for both skins and core.) In both cases, the solution of practical problems often demands the application of computational techniques. Among the computational techniques, the finite element method (FEM) has assumed a significant role in both academic and industrial environments. In this context, three-dimensional analysis can become very expensive when complex structures are modeled. Computational costs are greatly reduced when two-dimensional models are implemented, especially in those analyzes in which the whole sandwich (core and skins) is modeled as a unique equivalent plate. An increase of computational costs is introduced by those analyses in which the sandwich plate is modeled as three independent plates: the two skins and a core. Following Reddy,[1] it is intended that those theories that preserve the number of variables independent of the number of layers are herein denotes as equivalent single-layer models (ESLM), whereas those two-dimensional theories in which the same variables are independent in each layer are denoted as layerwise models (LWM).

Many papers have been published on modeling sandwich plates. Classical and refined plate theories in both the ESLM and LWM frameworks have been employed. Among the overview papers is one by Noor et al.[2] However, most of the conducted analyses have the following peculiarities: 1) They are restricted to academic sandwich plate problems that are related to simply supported sandwich plates loaded by harmonic transverse pressure, such as that by Pagano.[3] 2) Numerical results and comparisons are given for few classical and/or refined plate models. Interest in additional benchmarking has been shown by recent articles by Meyer-Piening and Stefanelli[4] and Meyer-Piening,[5] who extended the method in Ref. 3 to the case of locally loaded rectangular, simply supported sandwich plate.

The present work aims to contribute to these two points by considering additional benchmark problems, as well as by enlarging the number of the theories considered in the numerical investigations. Recent authors' findings[6–9] are herein extended. Classical modelings based on displacement assumptions as well as mixed theories that assume both displacement and transverse stress variables have been developed and compared in Refs. 6–9. Reissner's mixed variational theorem (RMVT) and the principle of virtual displacements (PVD) are used to derive governing equations. LWM and ESLM are developed including and/or discarding so-called zigzag (ZZ) and interlaminar continuity (IC). Linear up-to-fourth-order thickness expansions are discussed, and closed-form as well as FE implementations are presented. The availability of this large amount of modeling in conjunction with the treatment of early benchmark problems by Pagano[3] (BM1) and recent benchmark problems by Meyer-Piening[5] (BM2) can produce an exhaustive assessment of available two-dimensional theories for sandwich composites plates.

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Mechanics of Two-Dimensional Modelings

Two-dimensional models of sandwich plates are developed by making assumptions in the thickness plate directions z. Displacements $\boldsymbol{u}$ and transverse stresses $\boldsymbol{\sigma}_n$ are the variables expanded. Such expansions are made according to the following formulas:

$$\boldsymbol{u} = F_t\boldsymbol{u}_t + F_b\boldsymbol{u}_b + F_r\boldsymbol{u}_r = F_\tau\boldsymbol{u}_\tau$$

$$\tau = t,\ b,\ r, \qquad r = 2, \ldots, N \quad (1)$$

$$\boldsymbol{\sigma}_n = F_t\boldsymbol{\sigma}_{n_t} + F_b\boldsymbol{\sigma}_{n_b} + F_r\boldsymbol{\sigma}_{n_r} = F_\tau\boldsymbol{\sigma}_{n_\tau}$$

$$\tau = t,\ b,\ r, \qquad r = 2, \ldots, N \quad (2)$$

Boldface letters denote arrays ($\boldsymbol{u} = \{u_x,\ u_y,\ u_z\}$, $\boldsymbol{\sigma}_n = \{\sigma_{n_x},\ \sigma_{n_y},\ \sigma_{n_z}\}$, and so on). F_t, F_b, and F_r are the base functions used for the z expansions, and the first two polynomials are related to the the linear part of such expansions, whereas F_r introduces the $N-1$ higher-order terms. (The power of z and Legendre polynomials are used to build up F_t, F_b, and F_r.) The same meaning is assumed by the related introduced variables $\boldsymbol{u}_t$, $\boldsymbol{u}_b$, $\boldsymbol{u}_r$, $\boldsymbol{\sigma}_{nt}$, $\boldsymbol{\sigma}_{nb}$, and $\boldsymbol{\sigma}_{nr}$.

Governing equations in strong and weak form of the introduced two-dimensional models are written according to two variational statements: PVD and RMVT.[6,10,11] The first one, which states, in the static case,

$$\int_V \left(\delta\boldsymbol{\epsilon}_{pG}^T\boldsymbol{\sigma}_{pH} + \delta\boldsymbol{\epsilon}_{nG}^T\boldsymbol{\sigma}_{nH}\right)\mathrm{d}V = \delta L_e \quad (3)$$

is used to derive governing equations if only displacement assumptions are made. The superscript T signifies an array transposition, whereas the subscript p denotes in-plane components $\boldsymbol{\sigma}_p = \{\sigma_{xx},\ \sigma_{yy},\ \sigma_{xy}\}$ and $\boldsymbol{\epsilon}_p = \{\epsilon_{xx},\ \epsilon_{yy},\ \epsilon_{xy}\}$. The subscript H denotes that stresses are computed via Hooke's law. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress from Hooke's law and strain from geometrical relations (subscript G). Here δL_e is the virtual variation of the work made by the external layer force $\boldsymbol{p}$.

RMVT, which states

$$\int_V \left[\delta\boldsymbol{\epsilon}_{pG}^T\boldsymbol{\sigma}_{pH} + \delta\boldsymbol{\epsilon}_{nG}^T\boldsymbol{\sigma}_{nM} + \delta\boldsymbol{\sigma}_{nM}^T(\boldsymbol{\epsilon}_{nG} - \boldsymbol{\epsilon}_{nH})\right]\mathrm{d}V = \delta L_e \quad (4)$$

is employed in both assumed displacement and stress variables. The third mixed term variationally enforces the compatibility of the transverse strain components. Subscript M denotes that transverse stresses are those of the assumed model. As far as Hooke's law and geometrical relations are concerned, reference is made to the formulas given in Ref. 7.

Implemented Theories

The thickness assumptions made for Eqs. (1) and (2) permit one to develop a large variety of two-dimensional theories. Depending on the variational statement used (PVD or RMVT), the description of the variables (LWM or ESLM), and the order of the expansion used, N, a number of two-dimensional theories can be constructed. Such a variety of sandwich theories fits very well with the assessment proposed in this paper. In fact, these theories are able to cover a large part of the known classical and refined modelings of sandwich plates. The richest one, LWM based on PVD with $N = 4$ (LM4), leads to a quasi-three-dimensioanl description of sandwich plates; the poorest, ESLM based on PVD with $N = 1$ (ED1), leads to results very close to Kirchhoff-type approximation theories (see Refs. 6–9). A few details on the assumptions related to the different plate theories is given next. Details may be found in Refs. 6–9.

Plate Theories with Only Displacement Variables

First are ESLM classical models based on PVD with $N = 1$–4 (ED1–ED4). The Taylor-type expansion is used for the displacement of the whole plate. This is written in the following unified notation: Subscript b denotes values with correspondence to Ω ($\boldsymbol{u}_b = \boldsymbol{u}_0$)

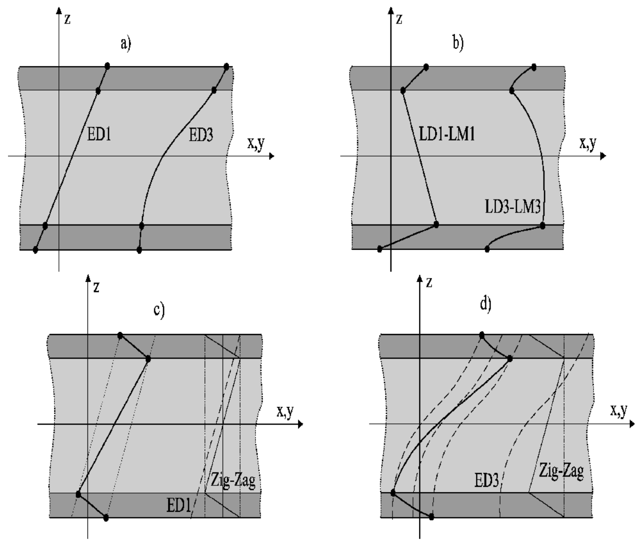

Fig. 1 Examples of ESL and LW assumptions: a) linear and cubic ESL cases, b) linear and cubic LW cases, c) linear case for ESL assumptions and d) cubic case for ESL assumptions including ZZ functions.

whereas subscript t refers to the highest-order term ($\boldsymbol{u}_t = \boldsymbol{u}_N$). The F_τ polynomials assume the following explicit form: $F_b = 1$, $F_t = z^N$, and $F_r = z^r$, where $r = 2, \ldots, N-1$.

Second are ESLM classical models based on PVD with $N = 1$–3 including Murakami's[11] ZZ functions $(-1)^k 2z_k/h_k$ (EDZ1–EDZ3). It is possible to introduce ZZ effects in the preceding expansion and in the PVD framework by referring to Murakami's idea, which was originally introduced in the framework of RMVT. Murakami proposed to add a ZZ-type function in a Taylor-type expansion. According to our notation, one can assume $F_t = (-1)^k\zeta_k$. (Here $\zeta_k = 2z_k/h_k$ is a nondimensional layer coordinate, where z_k is the physical coordinate of the k layer the thickness of which is h_k.) The exponent k changes the sign of the ZZ term in each layer. This permits one to reproduce the discontinuity of the first derivative of the displacement variables in the z directions, which physically comes from the intrinsic transverse anisotropy of multilayered structures (Fig. 1).

Third are LWM based on PVD with $N = 1$–4 (LD1–LD4). An LW description is simply obtained by assuming the displacement expansion of the preceding section in each layer. Nevertheless, Taylor-type expansion is not convenient for an LW description. In fact, the fulfillment of continuity requirements for the displacement at interfaces could be easily introduced by using the interface variables as unknowns. A convenient combination of Legendre polynomials[6] could be used as base functions. The subscripts t and b denote values related to the layer top and bottom surface, respectively. These two terms consist of the linear part of the expansion. The thickness functions $F_\tau(\zeta_k)$ have now been defined at the k-layer level, $F_t = (P_0 + P_1)/2$, $F_b = (P_0 - P_1)/2$, and $F_r = P_r - P_{r-2}$, where $r = 2, 3, \ldots, N$ in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the j order defined in the ζ_k domain: $-1 \le \zeta_k \le 1$.

Theories with Displacement and Transverse Stress Variables

The first case is ESLM based on RMVT with $N = 1$–4 fulfilling IC (EMC1–EMC4). Taylor-type expansion is not appropriate for an ESL description of transverse stresses. Its use would require additional constrains to fulfill transverse shear and normals stress continuity. The use of RMVT demands an LW description of transverse stresses even though ESLM expansions are used for displacements. (It is intended that in the presented derivations the ESLM description is only related to displacement fields in RMVT applications.) The LWM already used for displacements is extended to transverse stresses. Taylor expansion is instead preserved for displacement variables.

The second case is ESLM based on RMVT with $N = 1$–4 fulfilling ZZ and IC (EMZC1–EMZC3). In this case the Murakamsi's ZZ function[11] is used in the displacement expansion related to EMC1–EMC3.

The last case is LWM based on PVD with $N = 1$–4 (LM1–LM4). A full LW description can be introduced by simply extending the LW description to both transverse stress and displacement variables. Note that the LW description does not require a Murakami's function[11] for the simulation of ZZ effects.

Closed-Form Solutions

On substitution of introduced displacement and stress models, as well as a suitable geometric relation and Hooke's law, and by the integration by parts, the two described variational equations, PVD and RMVT, lead to governing differential equations in terms of the introduced stress and displacement variables. The displacement formulation yields the following equilibrium conditions:

$$\delta \boldsymbol{u}_\tau^k : \boldsymbol{K}_d^{k\tau s} \boldsymbol{u}_s^k = \boldsymbol{p}_\tau^k \tag{5}$$

The related boundary conditions are

$$\boldsymbol{u}_\tau^k = \bar{\boldsymbol{u}}_\tau^k \qquad \text{or} \qquad \boldsymbol{\Pi}_d^{k\tau s} \boldsymbol{u}_s^k = \boldsymbol{\Pi}_d^{k\tau s} \bar{\boldsymbol{u}}_s^k \tag{6}$$

The mixed case leads to the following set of equilibrium and constitutive equations:

$$\begin{aligned} \delta \boldsymbol{u}_\tau^k &: \boldsymbol{K}_{uu}^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{K}_{u\sigma}^{k\tau s} \boldsymbol{\sigma}_{ns}^k = \boldsymbol{p}_\tau^k \\ \delta \boldsymbol{\sigma}_{n\tau}^k &: \boldsymbol{K}_{\sigma u}^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{K}_{\sigma\sigma}^{k\tau s} \boldsymbol{\sigma}_{ns}^k = 0 \end{aligned} \tag{7}$$

and to the boundary conditions

$$\boldsymbol{u}_\tau^k = \bar{\boldsymbol{u}}_\tau^k \qquad \text{or} \qquad \boldsymbol{\Pi}_u^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \boldsymbol{\sigma}_{ns}^k = \boldsymbol{\Pi}_u^{k\tau s} \bar{\boldsymbol{u}}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \bar{\boldsymbol{\sigma}}_{ns}^k \tag{8}$$

The additional subscript/superscript $s = t$, b, and r has been introduced to distinguish the terms related to the introduced variables from those related to their variations. The explicit form of the introduced arrays along with the manner to build multilayered equations have been described in Refs. 6–9.

The obtained boundary-values problem governed in the most general case of geometry, boundary conditions, and layouts could be solved by implementing only approximated solution procedures. In the particular case in which the material has orthotropic behavior, Navier-type closed-form solutions can be found by assuming the harmonic forms for the applied loadings and unknown variables.[7]

FEM Equations

The assumed displacement field is first introduced in the expression for the strains; second, finite element approximations are used to express the displacement in terms of their nodal values, via shape functions,

$$\boldsymbol{u}_\tau^k = N_i \boldsymbol{q}_{\tau i}^k, \qquad i = 1, 2, \ldots, N_n \tag{9}$$

where N_n is the number of the FE nodes in the element and $\boldsymbol{q}_{\tau i}^k = q_{u_x \tau i}^k, q_{u_y \tau i}^k$, and $q_{u_z \tau i}^k$ are the nodal variables. The finite element equations can be written as

$$\delta \boldsymbol{q}_{\tau i}^{kT} : \boldsymbol{K}^{k\tau s i j} \boldsymbol{q}_{sj}^k = \boldsymbol{P}_{\tau i}^k \tag{10}$$

The explicit form may be found in Refs. 8 and 9. When N and N_n are varied, the finite element matrices of the k layer, corresponding to the implemented two-dimensional theories and number of nodes, are obtained.

For RMVT applications, transverse stress variables are expressed in terms of shape functions as done for the displacements,

$$\boldsymbol{\sigma}_{n\tau}^k = N_i \boldsymbol{g}_{\tau i}^k, \qquad i = 1, 2, \ldots, N_n \tag{11}$$

where $\boldsymbol{g}_{\tau i}^k = g_{xz\tau i}^k, g_{yz\tau i}^k$, and $g_{zz\tau i}^k$ are the nodal stress variables. When the definition of virtual variations is imposed, the RMVT leads to the following equilibrium and compatibility equations:

$$\begin{aligned} \delta \boldsymbol{q}_{\tau i}^{kT} &: \boldsymbol{K}_{uu}^{k\tau s i j} \boldsymbol{q}_{sj}^k + \boldsymbol{K}_{u\sigma}^{k\tau s i j} \boldsymbol{g}_{sj}^k = \boldsymbol{P}_{\tau i}^k \\ \delta \boldsymbol{g}_{\tau i}^{kT} &: \boldsymbol{K}_{\sigma u}^{k\tau s i j} \boldsymbol{q}_{sj}^k + \boldsymbol{K}_{\sigma\sigma}^{k\tau s i j} \boldsymbol{g}_{sj}^k = \boldsymbol{0} \end{aligned} \tag{12}$$

Explicit forms are given in Ref. 8.

Benchmarks Description

BM1 consists of a square sandwich plate bent by a transverse with bisinusoidal distribution of transverse pressure applied at the top plate surface (Fig. 2a). Different values of the thickness parameter a/h are treated. The faces have equal thickness $h_f = 0.1h$. The mechanical properties of the laminas that are used as skins are $E_L = 25 \times 10^6$ psi (172,375 MPa), $E_T = 1 \times 10^6$ psi (6895 MPa), $G_{LT} = 0.5 \times 10^6$ psi (3516 MPa), $G_{TT} = 0.2 \times 10^6$ psi (1379 MPa), and $\nu_{LT} = \nu_{TT} = 0.25$, where, following the usual notations, L is the fiber direction, T is transverse direction, and ν_{LT} is the major Poisson ratio. The core material used for the sandwich plates is transversely isotropic with respect to the z axis and is characterized by the following elastic properties: $E_{xx} = E_{yy} = 0.04 \times 10^6$ psi (275.8 MPa), $E_{zz} = 0.5 \times 10^6$ psi (3447.5 MPa), $G_{xz} = G_{yz} = 6 \times 10^4$ psi (413.7 MPa), $G_{xy} = 1.6 \times 10^4$ psi (1103.2 MPa), and $\nu_{xy} = \nu_{zy} = \nu_{zx} = 0.25$. The principal material directions of the core always coincide with geometrical axes of the plate. Stress and displacement values have been normalized according to the relations $(\bar{\sigma}_{xx}, \bar{\sigma}_{yy}) = [1/p_z(a/h)^2](\sigma_{xx}, \sigma_{yy})$, $(\bar{\sigma}_{zx}, \bar{\sigma}_{zy}) = [1/p_z(a/h)](\sigma_{zx}, \sigma_{zy})$, $(\bar{U}_x) = (u_x)(E_T h/p_z a)$, and $\bar{U}_z = u_z[100 \cdot h^3 \cdot E_T/p_z \cdot a^4]$, where p_z is the amplitude of transverse applied pressure. Pagano[3] provided a three-dimensional solution for this problem in the case of simply supported edges. Closed-form solutions are available in this case. This benchmark has been used as a reference solution by many authors. It will be denoted as BM1a. The additional case of clamped edges is discussed and referenced as BM1b. This last case has only been investigated by FE analysis. Therefore, BM1b could be useful to assess approximated solutions related to two-dimensional modelings.

BM2 consists of an extension of the BM1 to the case of localized loadings of thin sandwich rectangular plates. It will essentially show that accurate two-dimensional analyses can be required even though a thin sandwich structure is analyzed. Such a requirement is due to the presence of local phenomena that come as a consequences of the application of a transverse pressure in a small area of the top surface. BM2 was proposed by Meyer-Piening,[5] who also provided a three-dimensional analytical solution and compared these results to FE analysis conducted by means of commercial code Nastran. (Three-dimensional brick elements were used for the core and plate elements for the two faces.) This is a sandwich, thin, rectangular plate loaded by a transverse pressure applied in a small rectangular region located at the plate centers. The plates are simply supported with correspondence to their four edges. The plate geometrical parameters hold: $a = 100$ mm, $b = 200$ mm, and $h = 12$ mm. The faces, of same material, have different thicknesses: top face thickness $h_3 = 0.1$ mm and bottom face thickness $h_1 = 0.5$ mm. The core thickness is $h_1 = 11.4$ mm.

The load consists of a transverse pressure applied of 1 MPa applied in a rectangular zone located at the plate center whose dimensions are 5×20 mm (Fig. 2b). Such a loading situation is very common in practice. In fact, it happens every time a concentrate loading is applied to a sandwich structure. The two faces have the following properties: $E_x = 70{,}000$ MPa, $E_y = 71{,}000$ MPa, $E_z = 69{,}000$ MPa, $E_{xz} = 26{,}000$ MPa, $E_{yz} = 26{,}000$ MPa, $E_{yz} = 26{,}000$ MPa, and $\nu_{xz} = \nu_{yz} = \nu_{yz} = 0.3$. The core consists of metallic foam that has the following properties: $E_x = E_y = 3$ MPa, $E_z = 2.8$ MPa, $E_{xz} = E_{yz} = E_{yz} = 1$ MPa, and $\nu_{xz} = \nu_{yz} = \nu_{yz} = 0.25$.

Results and Discussion

A comprehensive numerical analysis has been conducted to compare analytical as well as FE solutions described in the second section. More than 20 plate theories and related FEs have been compared. In-plane and out-of-plane displacement and stress components have been investigated. Transverse shear and normal stresses have been computed by postprocessing of three-dimensional equilibrium equations. Transverse normal stress σ_{zz} and strain ϵ_{zz} have always been retained. Readers interested in an explicit evaluation of transverse stress/strain effects may refer to Ref. 7.

BM1a results are given in Tables 1 and 2. FE solutions concerning displacement and stresses have been compared for classical and advanced mixed theories in both ESL and LW framework. If available,

Table 1 Comparison of in-plane stress $\bar{\sigma}_{xx}(a/2,\ a/2,\ \pm 1/2h)$ from different two-dimensional theories for problem BM1a

	$a/h = 2$		$a/h = 10$		$a/h = 50$	
Theory	$h/2$	$-h/2$	$h/2$	$-h/2$	$h/2$	$-h/2$
Three-dimensional analysis	3.278	−2.653	1.153	−1.152	1.099	−1.099
			LW analyses			
LM4[a]	3.2793	−2.6543	1.1536	−1.1517	1.0989	−1.0989
LM4	3.2426	−2.6233	1.1323	−1.1293	1.0786	−1.0786
LM3	3.2415	−2.6224	1.1324	−1.1293	1.0786	−1.0786
LM2	3.2352	−2.6172	1.1323	−1.1293	1.0785	−1.0785
LM1	3.0867	−2.4860	1.1332	−1.1304	1.0792	−1.0792
LD4[a]	3.2810	−2.6555	1.1536	−1.1517	1.0989	−1.0989
LD4	3.2420	−2.6228	1.1323	−1.1293	1.0785	−1.0785
LD3	3.2426	−2.6233	1.1324	−1.1293	1.0785	−1.0785
LD2	3.2259	−2.6091	1.1322	−1.1296	1.0785	−1.0785
LD1	3.0917	−2.5041	1.1297	−1.1270	1.0792	−1.0791
			ESL analyses			
EMZC3[a]	3.2295	−2.5319	1.1501	−1.1465	1.0975	−1.0974
EMZC3	3.1594	−2.5612	1.1488	−1.1467	1.0714	−1.0714
EMZC2	3.1255	−2.5286	1.1588	−1.1568	1.0843	−1.0843
EMZC1	2.9179	−2.8960	1.1454	−1.1445	1.0690	−1.0689
EDZ3[a]	3.2487	−2.5486	1.1502	−1.1465	1.0973	−1.0972
EDZ3	3.1623	−2.5617	1.1484	−1.1462	1.0712	−1.0711
EDZ2	3.1331	−2.5323	1.1465	−1.1443	1.0710	−1.0710
EDZ1	2.9224	−2.9011	1.1427	−1.1418	1.0678	−1.0678
EMC4	2.9766	−2.4009	1.1435	−1.1424	1.0697	−1.0697
EMC3	3.0610	−2.4874	1.1444	−1.1427	1.0710	−1.0710
EMC2	1.0138	−0.47231	1.0455	−1.0425	1.0642	−1.0641
EMC1	0.81337	−0.76710	1.0419	−1.0399	1.0629	−1.0628
ED4[a]	3.1442	−2.5407	1.1452	−1.1438	1.0960	−1.0960
ED4	2.9828	−2.4080	1.1440	−1.1430	1.0695	−1.0695
ED3	3.0752	−2.5015	1.1452	−1.1436	1.0706	−1.0706
ED2	1.0181	−0.47943	1.0469	−1.0441	1.0635	−1.0634
ED1	0.83025	−0.79552	1.0446	−1.0431	1.0604	−1.0603
FSDT[a]	1.0921	−1.0921	1.0484	−1.0484	1.0902	−1.0902
CLT[a]	0.7555	−0.7555	1.0921	−1.0921	1.0921	−1.0921

[a]Analytical closed-form solutions.

Table 2 Comparison of transverse shear stress $\bar{\sigma}_{zx}(0,\ a/2,\ 0)$ from different two-dimensional theories for problem BM1a

	a/h				
Theory	2	4	10	20	50
Three-dimensional analysis	0.186	0.239	0.3	0.317	0.323
		LW analyses			
LM4[a]	0.18435	0.23860	0.29978	0.31736	0.32313
LM4	0.18976	0.24593	0.30425	0.32225	0.32872
LM3	0.15957	0.24583	0.30425	0.32225	0.32372
LM2	0.18533	0.24352	0.30323	0.32148	0.32810
LM1	0.19039	0.24533	0.30350	0.32160	0.32810
LD4[a]	0.18435	0.23860	0.29978	0.31736	0.32313
LD4	0.17846	0.22778	0.28019	0.29595	0.30177
LD3	0.17846	0.22778	0.28019	0.29595	0.30177
LD2	0.17834	0.22778	0.28019	0.29595	0.30177
LD1	0.17855	0.22775	0.28007	0.29579	0.30162
		ESL analyses			
EMZC3[a]	0.18343	0.23795	0.29918	0.31687	0.32268
EDZ3[a]	0.18301	0.23751	0.29905	0.31678	0.32261
ED4[a]	0.18807	0.24353	0.30185	0.31792	0.32314
FSDT[a]	0.25063	0.28351	0.31339	0.32024	0.32234
CLT[a]	0.32275	0.32275	0.32275	0.32275	0.32275

[a]Analytical closed-form solutions.

the three-dimensional solutions by Pagano[3] have been given as the first row of Tables 1 and 2. Analytical closed-form solutions are also given in some cases. Differences between FE and analytical solutions must be referenced to the finite number of elements that have been used in the FE model of the plate. A mesh of 5×5 of nine-noded $Q9$ plate elements have been used. Table 1 shows a comparison of transverse displacements located at the plate center. There are 8 LW analyzes compared to 14 ESL results, classical lamination theory (CLT), and first-order shear deformation theory (FSDT). Therefore, 24 theories are presented. Very thick ($a/h = 2$), moderately thick ($a/h = 10$), and thin ($a/h = 50$) sandwiches have been investigated. The following results are notable:

1) The good agreement between FE and analytical solutions proves the effectiveness of the FE model used.

2) LW analysis is very effective for thick and very thick plates in both case of classical and mixed theories. Good results have

Table 3 Comparison of transverse displacement $\bar{U}_z$ ($a/2, a/2, 0$) from different theories for problem BM1b

	a/h		
Theory	2	10	100
	LW analyses		
LM4	17.231	1.4405	0.22573
LM1	17.690	1.4407	0.22573
LD4	17.222	1.4403	0.22573
LD1	17.426	1.4365	0.22569
	ESL analyses		
EMZC3	17.158	1.4326	0.22520
EDZ3	17.115	1.4319	0.22518
EMC4	16.387	1.3530	0.22414
ED4	16.243	1.3445	0.22400
ED1	11.448	0.79966	0.21487

Table 4 Comparison of transverse shear stress $\bar{\sigma}_{zx}(0, a/2, 0)$ from different theories for problem BM1b

Theory	2	10	100
	Present LW analyses		
LM4	0.10587	0.22800	0.36012
LM1	0.10322	0.22630	0.36061
LD4	0.09896	0.22898	0.36014
LD1	0.08571	0.22499	0.36013
	ESL analyses		
EMZC3	0.07295	0.21750	0.34863
EDZ3	0.07512	0.21743	0.34864
EMC4	0.09529	0.23261	0.35104
ED4	0.09570	0.23294	0.35104
ED1	0.23134	0.31634	0.34521

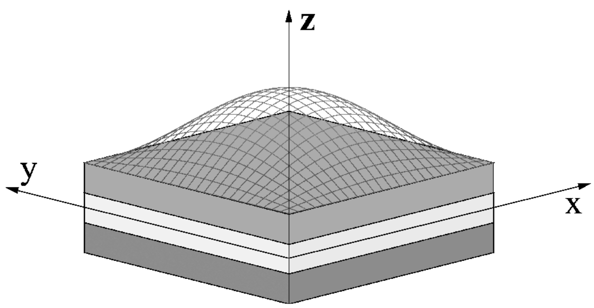

Fig. 2a Pagano[3] sandwich square plates subjected to bisinusoidal distribution of transverse pressure (problem BM1).

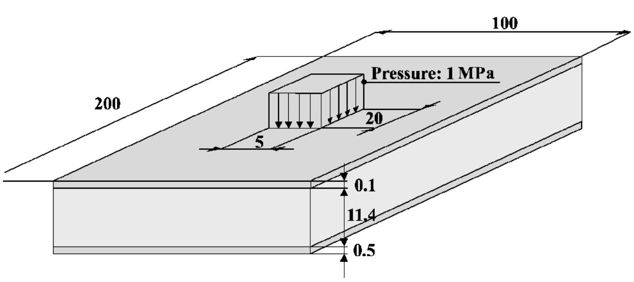

Fig. 2b Meyer-Piening[5] rectangular sandwich plates subjected to transverse pressure located at the plate center (problem BM2).

been obtained even though linear distribution in the layer-thickness direction have been adopted (LM1 and LD1).

3) The accuracy of equivalent single-layer models is very subordinate to the thickness parameter, to order of the expansion of the variables in the thickness direction, as well as to the adopted formulation (classical or mixed).

4) Results related to the whole modelings merge in the thin sandwich response. Quite different responses have been found in thick plate cases.

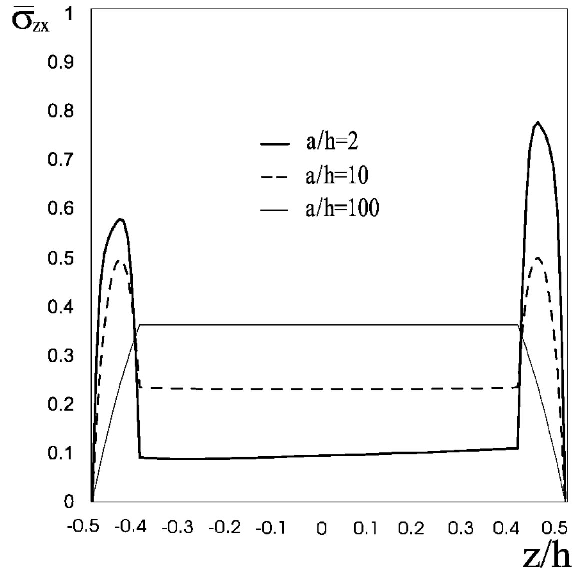

Fig. 3a Clamped edges case $\bar{\sigma}_{zx}$ vs z/h for BM1b from LM4 analysis.

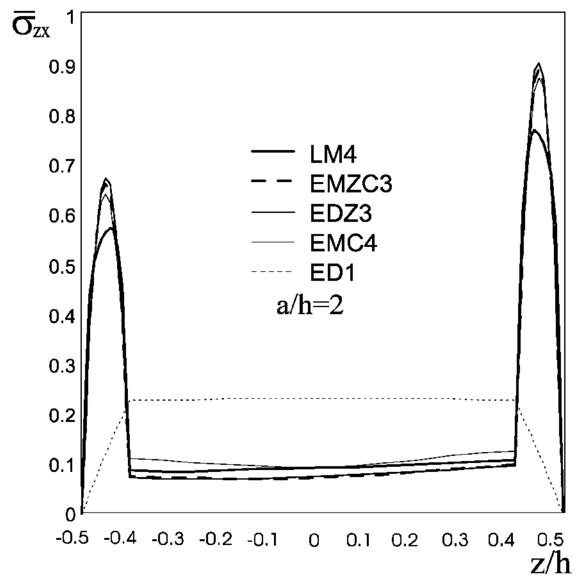

Fig. 3b Comparison of $\bar{\sigma}_{zx}$ vs z/h response from five theories for problem BM1b.

5) RMVT becomes very effective as far as the ESLM description is concerned: EMZC1–EMZC3 analyses lead to much better descriptions than ED1–ED3 analyses.

6) The comparison between ED1–ED4 and EDZ1–EDZ4 analyses show that the results related to ED1–ED4 theories are very improved by the inclusion of ZZ functions. One could conclude that the ZZ function is more effective than the power of z polynomials.

7) Classical ESLM analyses based on PVD provide the most inferior descriptions. In particular, theories with linear through-the-thickness displacement field (ED1) are inaccurate even when moderately thick sandwich plates are considered.

8) As far as FE results are concerned, the number of elements of LW analysis is three times larger than corresponding ESLM results. In fact, LW analyses are closer than ESLM analyses with respect to corresponding analytical solutions. The same analyses have been conducted in Table 2, where transverse shear stresses are compared. The earlier comments have been confirmed for stress evaluations. Larger discrepancies among theories exist with stress evaluations as opposed to displacement evautaions.

Results related to clamped edges cases are presented in Tables 3 and 4 and Figs. 3a and 3b. A finer mesh of 10×10 of $Q9$ plate elements has been used in this case. Results on convergence rates

Table 5 Comparison of different analyses for problem BM2

Analysis	z	U_z, mm	σ_{yy}, MPa	σ_{xx}, MPa	σ_{xy}, MPa
		Upper face			
Three-dimensional analytical[5]	Top	−3.78	−241	−624	0
	Bottom		211	580	0
Three-dimensional NASTRAN[5]	Top	−3.84	−237	−628	0
	Bottom		102	582	0
Two-dimensional LM2 present	Top	−3.7628	−223.93	−595.56	0
	Bottom		196.37	556.00	0
Two-dimensional EMZC3 present	Top	−2.0483	−122.59	−214.27	0
	Bottom		99.62	181.40	0
Two-dimensional ED1 present	Top	−0.0187	−23.99	−29.46	0
	Bottom		−23.75	−29.17	0
		Lower face			
Three-dimensional analytical[5]	Top		−121	−138	0
	Bottom	−2.14	127	146	0
Three-dimensional NASTRAN[5]	Top		−120	−140	0
	Bottom	−2.19	127	148	0
Two-dimensional LM2 present	Top		−118.99	−136.20	0
	Bottom	−2.1403	125.00	144.03	0
Two-dimensional EMZC3 present	Top		−190.7	−227.67	0
	Bottom	−1.8717	191.11	230.76	0
Two-dimensional ED1 present	Top		3.32	4.87	0
	Bottom	−0.0181	4.50	6.36	0

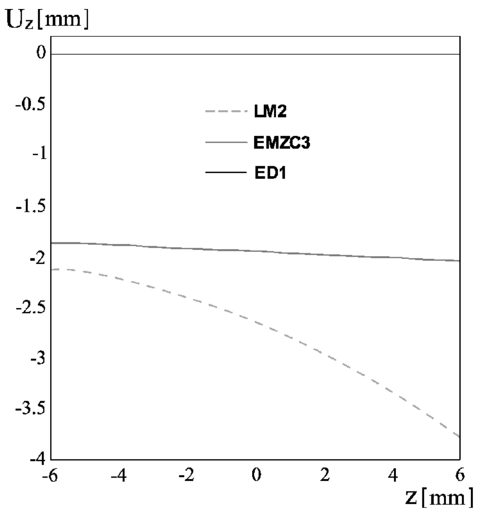

Fig. 4 Comparison of U_z vs z response from three theories for problem BM2.

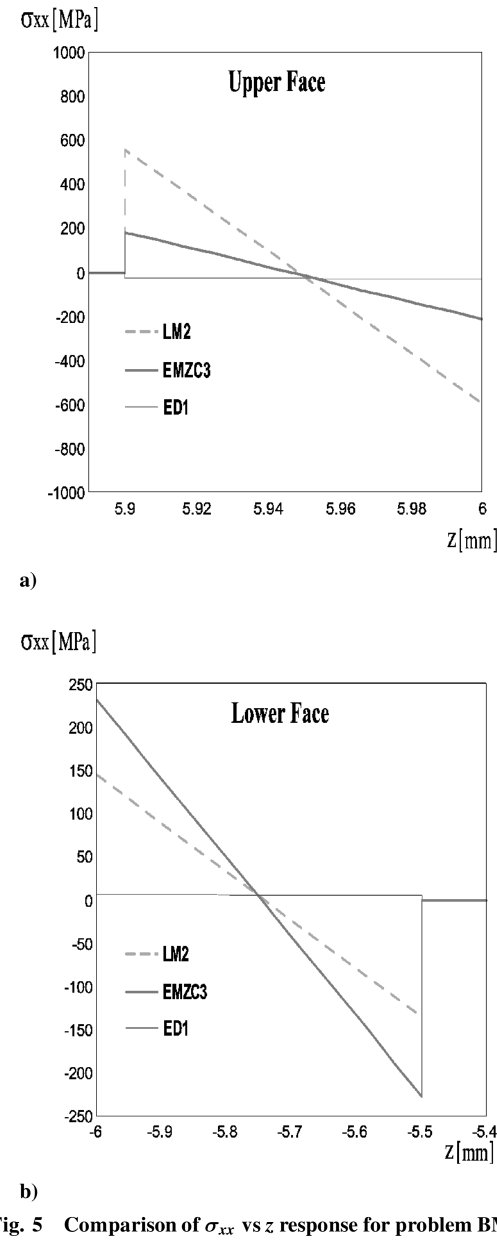

Fig. 5 Comparison of σ_{xx} vs z response for problem BM2.

have been omitted for brevity. Only results related to nine of the most significant theories are presented in Tables 3 and 4. Three-dimensional results are not available in this case. Nevertheless, BM1a results, as well as previous work,[7] could lead to reference the LM4 results as a three-dimensional description of displacement and stress states for the considered clamped sandwich plates. The comments made for BM1a are confirmed. Figures 3a and 3b show transverse shear stress distributions in the core and face thickness z direction. Some of the most significant theories and values of the thickness parameters are considered. These diagrams are extremely significant. They show that the differences between a certain theory and the LM4 results are very subordinate to the value of the z coordinate. In particular, transverse shear stresses are much better evaluated in the core with respect to the evaluations given in the two faces. Note the unsymmetric distribution of τ_{zx} in very thick plate cases. Such asymmetry is due to σ_{zz} effects and cannot be described by ESL analyzes.[7]

Results for BM2 are shown in Figs. 4 and 5 and Table 5. Stresses and displacements are located at the plate center. Transverse displacements are considered in Fig. 4. In-plane stresses are detailed in Figs. 5a and 5b for the two sandwich faces. Three significant theories are compared: LM2, which leads to a three-dimensional description; EMZC3, which according to BM1 results consists of the best ESL description of a sandwich plates; and ED1, which is the closest to classical thin sandwich theory. Table 5 is a comparison of displacement and in-plane stresses with correspondence to the top/bottom points of the two faces. The results of present LM2, EMZC3, and ED1 analyses are compared to three-dimensional analytical and three-dimensional NASTRAN results reported in Ref. 5. The main comments on the presented results follow:

1) Table 5 shows that LM2 results are in excellent agreement with the three-dimensional solutions. In other words, LM2 consists of two-dimensional theories able to describe a complete stress field for BM2.

2) Even though a thin sandwich has been considered, both the refined EMZC3 and the classical ED1 ESL analyses are ineffective to trace the response of the BM2 problems.

3) The BM1 analyzes of the Meyer-Piening[5] test case has mostly shown that equivalent single-layer modelings are ineffective to capture the local strain/stress field caused by localized pressure even when a thin plate is considered.

Conclusions

This paper has presented an assessment of two-dimensional theories for sandwich plate analysis. Classical and mixed formulations have been implemented in both frameworks of LW and ESL theories. FE and closed-form solutions have been considered. Results have been presented in the form of tables and diagrams for two benchmark problems. The considered benchmarks demonstrate both the capability and limitations of the considered two-dimensional theories. The convenience of using LW descriptions and advanced mixed theories has been demonstrated. The second benchmark has especially shown that the use of LWM is mandatory to capture the local effects related to the localized application of transverse pressure even though thin sandwich plates are considered.

The large variety of two-dimensional modelings considered for BM1 along with the local effects addressed by BM2 could suggest test beds to assess other two-dimensional modelings that have not been considered in this work or are proposed for future works.

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